

Comparison results for nonlinear anisotropic parabolic problems

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Abstract

Comparison results for solutions to the Dirichlet problems for a class of nonlinear, anisotropic parabolic equations are established. These results are obtained through a semi-discretization method in time after providing estimates for solutions to anisotropic elliptic problems with zero-order terms.

1 Introduction

In this work we prove comparison results for a class of nonlinear anisotropic parabolic problems whose model case is

$$\begin{cases} \partial_t u - \sum_{i=1}^N \left(\alpha_i |\partial_{x_i} u|^{p_i-2} \partial_{x_i} u \right)_{x_i} = f(x, t) & \text{in } Q_T := \Omega \times (0, T) \\ u(x, 0) = u_0(x) & \text{in } \Omega \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \end{cases} \quad (1.1)$$

where Ω is an open, bounded subset of \mathbb{R}^N with Lipschitz continuous boundary, $N \geq 2$, $T > 0$, $\alpha_i > 0$ and $p_i \geq 1$ for $i = 1, \dots, N$ such that their harmonic mean $\bar{p} > 1$ and the data f and u_0 have a suitable summability.

Problem (1.1) provides the mathematical models for natural phenomena in biology and fluid mechanics. For example, they are the mathematical description of the dynamics of fluids in anisotropic media when the conductivities of the media are different in different directions. They also appear in biology as a model for the propagation of epidemic diseases in heterogeneous domains.

In the last years, anisotropic problems have been largely studied by many authors (see *e.g.* [ACh, BMS, dC, DFG, DF, FGK, FGL, FS, Gi, M]). The growing interest has led to an extensive investigation also for problems governed by fully anisotropic growth condition (see *e.g.* [A, AC, AdBF1, C1, C2]) and problems related to different type of anisotropy (see *e.g.* [AFTL, BFK, DdB2, DG]).

We emphasize that, when $p_i = p \neq 2$ for $i = 1, \dots, N$ the anisotropic diffusion operator in problem (1.1) coincides with the so-called pseudo-Laplacian operator, whereas when $p_i = 2$ for $i = 1, \dots, N$ it coincides with usual Laplacian.

Symmetrization methods in a priori estimates for solutions to isotropic parabolic problems were widely used (see *e.g.* [ALT], [B], [D], [FV], [MR], [V] and the bibliography starting with it).

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As in the isotropic setting (see *e.g.* [Ta1]), if w solves the stationary anisotropic problem

$$\begin{cases} -\sum_{i=1}^N \left(\alpha_i |w_{x_i}|^{p_i-2} w_{x_i} \right)_{x_i} = f(x) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

rearrangement methods allows to obtain a pointwise comparison result for w (see [C2]). Namely,

$$w^\star(x) \leq z(x) \quad \text{for a.e. } \Omega^\star, \quad (1.2)$$

where Ω^\star is the ball centered in the origin such that $|\Omega^\star| = |\Omega|$, w^\star is the symmetric rearrangement of a solution w to problem (1.1) and z is the radial solution to the following isotropic problem

$$\begin{cases} -\operatorname{div} \left(\Lambda |\nabla z|^{\bar{p}-2} \nabla z \right) = f^\star(x) & \text{in } \Omega^\star \\ z = 0 & \text{on } \partial\Omega^\star, \end{cases} \quad (1.3)$$

with Λ a suitable positive constant, \bar{p} the harmonic mean of exponents p_1, \dots, p_N and f^\star the symmetric decreasing rearrangement of f .

In the parabolic setting, the pointwise comparison (1.2) need not hold, nevertheless it is possible to prove for fixed $t \in (0, T)$, the following integral comparison result

$$\int_0^s u^*(\sigma, t) d\sigma \leq \int_0^s v^*(\sigma, t) d\sigma \quad \text{in } (0, |\Omega|), \quad (1.4)$$

where u^* and v^* are the decreasing rearrangement with respect to the space variable of the solution u to problem (1.1) and of the solution v to the following isotropic "symmetrized" problem

$$\begin{cases} v_t - \operatorname{div} \left(\Lambda |\nabla v|^{\bar{p}-2} \nabla v \right) = f^\star(x, t) & \text{in } Q_T^\star := \Omega^\star \times [0, T] \\ v(x, 0) = u_0^\star(x) & \text{in } \Omega^\star \\ v(x, t) = 0 & \text{on } \partial\Omega \times [0, T], \end{cases} \quad (1.5)$$

respectively. We stress that in contrast to the isotropic case not only the space domain and the data of problem (1.5) are symmetrized with respect to the space variable, but also the ellipticity condition is subject to an appropriate symmetrization. Indeed the diffusion operator in problem (1.5) is the isotropic \bar{p} -Laplacian.

In order to obtain the integral comparison result (1.4) we will use the method of semi-discretization in time. This approach was firstly used by ([V]) and ([ALT]) and consists into approximating the solution of a parabolic problem with a sequence of solutions to elliptic problems with zero-order terms. For this reason, we first prove an integral comparison result for such elliptic problems and then, passing to the limit, we obtain (1.4). We emphasize that integral comparison (1.4) implies a priori estimates for any Lorentz norm of $u(\cdot, t)$ in terms of the same norm of $v(\cdot, t)$ for any fixed $t > 0$. Moreover, we study the asymptotic behavior of solution $u(\cdot, t)$ as the time variable t goes to infinity. The paper is organized as follows. In Section 2 we recall some backgrounds on the anisotropic spaces and on the properties of symmetrization. In Section 3 we prove a integral comparison result for elliptic anisotropic problems and the main results.

2 Preliminaries

2.1 Anisotropic spaces

Let Ω be an open, bounded subset of \mathbb{R}^N with Lipschitz continuous boundary, $N \geq 2$, and let $1 \leq p_1, \dots, p_N < \infty$ be N real numbers. We define the anisotropic Sobolev space $W_0^{1,p_i}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,p_i}(\Omega)} = \|u\|_{L^1(\Omega)} + \|\partial_{x_i} u\|_{L^{p_i}(\Omega)}.$$

In this anisotropic setting, a *Poincaré-type inequality* holds (see [FGK]). If $u \in W_0^{1,p_i}(\Omega)$, for every $q \geq 1$ there exists a constant C , depending on $|\Omega|$ and q , such that

$$\|u\|_{L^q(\Omega)} \leq C \|\partial_{x_i} u\|_{L^q(\Omega)}. \quad (2.1)$$

We set $W_0^{1,\vec{p}}(\Omega) = \bigcap_{i=1}^N W_0^{1,p_i}(\Omega)$ with the norm

$$\|u\|_{W_0^{1,\vec{p}}(\Omega)} = \sum_{i=1}^N \|\partial_{x_i} u\|_{L^{p_i}(\Omega)} \quad (2.2)$$

and we denote its dual by $(W_0^{1,\vec{p}}(\Omega))'$.

Moreover we put $L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega)) = \bigcap_{i=1}^N L^{p_i}(0, T; W_0^{1,p_i}(\Omega))$ equipped with the following norm

$$\|u\|_{L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega))} = \sum_{i=1}^N \left(\int_0^T \|u_{x_i}\|_{L^{p_i}(\Omega)}^{p_i} dt \right)^{\frac{1}{p_i}}. \quad (2.3)$$

On denoting by \bar{p} the *harmonic mean* of p_1, \dots, p_N , i.e.

$$\frac{1}{\bar{p}} = \frac{1}{N} \sum_{i=1}^N \frac{1}{p_i}, \quad (2.4)$$

a Sobolev-type inequality tells us that whenever u belongs to $W_0^{1,\vec{p}}(\Omega)$, there exists a constant C_S such that

$$\|u\|_{L^q(\Omega)} \leq C_S \sum_{i=1}^N \|\partial_{x_i} u\|_{L^{p_i}(\Omega)} \quad (2.5)$$

where $q = \bar{p}^* = \frac{N\bar{p}}{N-\bar{p}}$ if $\bar{p} < N$ or $q \in [1, +\infty[$ if $\bar{p} \geq N$ (see [Tr]). If in plus $\bar{p} < N$, inequality (2.5) implies the continuous embedding of the space $W_0^{1,\vec{p}}(\Omega)$ into $L^q(\Omega)$ for every $q \in [1, \bar{p}^*]$. On the other hand, the continuity of the embedding $W_0^{1,\vec{p}}(\Omega) \subset L^{p_+}(\Omega)$ with $p_+ := \max\{p_1, \dots, p_N\}$ relies on inequality (2.1). It may happen that $\bar{p}^* < p_+$ if the exponents p_i are not closed enough. Then $p_\infty := \max\{\bar{p}^*, p_+\}$ turns out to be the critical exponent in the anisotropic Sobolev embedding.

2.2 Symmetrization

A precise statement of our results requires the use of classical notions of rearrangement and of suitable symmetrization of a Young function, introduced by Klimov in [K].

Let u be a measurable function (continued by 0 outside its domain) fulfilling

$$|\{x \in \mathbb{R}^N : |u(x)| > t\}| < +\infty \quad \text{for every } t > 0. \quad (2.6)$$

The *symmetric decreasing rearrangement* of u is the function $u^\star : \mathbb{R}^N \rightarrow [0, +\infty[$ satisfying

$$\{x \in \mathbb{R}^N : u^\star(x) > t\} = \{x \in \mathbb{R}^N : |u(x)| > t\}^\star \quad \text{for } t > 0. \quad (2.7)$$

The *decreasing rearrangement* u^* of u is defined as

$$u^*(s) = \sup\{t > 0 : \mu_u(t) > s\} \quad \text{for } s \geq 0,$$

where

$$\mu_u(t) = |\{x \in \Omega : |u(x)| > t\}| \quad \text{for } t \geq 0$$

denotes the *distribution function* of u .

Moreover

$$u^\star(x) = u^*(\omega_N |x|^N) \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Analogously, we define the *symmetric increasing rearrangement* u_\star on replacing “>” by “<” in the definitions of the sets in (2.6) and (2.7). Moreover, we set

$$u^{**}(s) = \frac{1}{s} \int_0^s u^*(r) dr \quad \text{for } s > 0.$$

We refer to [BS] for details on these topics.

We just recall the following property of rearrangements which will be useful in the following (see for example [ALT]):

Lemma 2.1 *If f, g are measurable functions defined in Ω , then*

$$\int_0^r (f + g)^*(s) ds \leq \int_0^r f^*(s) + g^*(s) ds, \quad \forall r \in [0, |\Omega|].$$

In this paper we will consider an N -dimensional Young function (namely an even convex function such that $\Phi(0) = 0$ and $\lim_{|\xi| \rightarrow +\infty} \Phi(\xi) = +\infty$) of the following type:

$$\Phi(\xi) = \sum_{i=1}^N \alpha_i |\xi_i|^{p_i} \quad \text{for } \xi \in \mathbb{R}^N \text{ with } \alpha_i > 0 \text{ for } i = 1, \dots, N. \quad (2.8)$$

We denote by $\Phi_\diamond : \mathbb{R} \rightarrow [0, +\infty[$ the symmetrization of Φ introduced in [K]. It is the one-dimensional Young function fulfilling

$$\Phi_\diamond(|\xi|) = \Phi_{\bullet\star\bullet}(\xi) \quad \text{for } \xi \in \mathbb{R}^N, \quad (2.9)$$

where Φ_\bullet is the Young conjugate function of Φ given by

$$\Phi_\bullet(\xi') = \sup \{ \xi \cdot \xi' - \Phi(\xi) : \xi \in \mathbb{R}^N \} \quad \text{for } \xi' \in \mathbb{R}^N.$$

So Φ_\diamond is the composition of Young conjugation, symmetric increasing rearrangement and Young conjugate again. Easy calculations show (see *e.g.* [C2]), that

$$\Phi_\diamond(|\xi|) = \Lambda |\xi|^{\bar{p}}, \quad (2.10)$$

where \bar{p} is the harmonic mean of exponents p_1, \dots, p_N defined in (2.4) and

$$\Lambda = \frac{2^{\bar{p}} (\bar{p} - 1)^{\bar{p}-1}}{\bar{p}^{\bar{p}}} \left[\frac{\prod_{i=1}^N p_i^{\frac{1}{p_i}} (p_i')^{\frac{1}{p_i'}} \Gamma(1 + 1/p_i')}{\omega_N \Gamma(1 + N/\bar{p}')} \right]^{\frac{\bar{p}}{N}} \left(\prod_{i=1}^N \alpha_i^{\frac{1}{p_i}} \right)^{\frac{\bar{p}}{N}} \quad (2.11)$$

with ω_N the measure of the N -dimensional unit ball, Γ the Gamma function and $p_i' = \frac{p_i}{p_i-1}$ with the usual conventions if $p_i = 1$.

We remember that in the anisotropic setting a *Polya-Szegö principle* holds (see [C2]). Let u be a weakly differentiable function in \mathbb{R}^N satisfying (2.6) and such that $\sum_{i=1}^N \alpha_i \int_{\mathbb{R}^N} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx < +\infty$, then u^\star is weakly differentiable in \mathbb{R}^N and

$$\Lambda \int_{\mathbb{R}^N} |\nabla u^\star|^{\bar{p}} dx \leq \sum_{i=1}^N \alpha_i \int_{\mathbb{R}^N} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx. \quad (2.12)$$

3 Main results

We deal with a class of nonlinear parabolic problems subject to general growth conditions and having the form

$$\begin{cases} u_t - \operatorname{div}(a(x, t, u, \nabla u)) = f(x, t) & \text{in } Q_T := \Omega \times (0, T) \\ u(x, 0) = u_0(x) & \text{in } \Omega \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, T), \end{cases} \quad (3.1)$$

where Ω is an open, bounded subset of \mathbb{R}^N with Lipschitz continuous boundary, $N \geq 2$, $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function such that, for *a.e.* $(x, t) \in Q_T$, for all $s \in \mathbb{R}$ and for all $\xi, \xi' \in \mathbb{R}^N$,

$$(H1) \quad a(x, t, s, \xi) \cdot \xi \geq \sum_{i=1}^N \alpha_i |\xi_i|^{p_i} \quad \text{with } \alpha_i > 0;$$

$$(H2) \quad |a_j(x, t, s, \xi)| \leq \beta \left[|s|^{\bar{p}/p_j'} + |\xi_j|^{p_j-1} \right] \quad \text{with } \beta > 0 \quad \forall j = 1, \dots, N;$$

$$(H3) \quad |a_j(x, t, s, \xi) - a_j(x, t, s', \xi)| \leq \gamma |\xi_j|^{p_j-1} |s - s'| \quad \text{with } \gamma > 0 \quad \forall j = 1, \dots, N;$$

$$(H4) \quad (a(x, t, s, \xi) - a(x, t, s, \xi')) \cdot (\xi - \xi') > 0 \quad \text{with } \xi \neq \xi'.$$

Moreover, we assume that

$$(H5) \quad f \in \sum_{i=1}^N L^{p_i'}(0, T, W^{-1, p_i'}(\Omega) + L^2(\Omega)) \quad \text{and} \quad u_0 \in L^2(\Omega).$$

Here, $1 \leq p_1, \dots, p_N < \infty$ and \bar{p} denotes the harmonic mean of p_1, \dots, p_N , defined in (2.4), such that $\bar{p} > 1$.

Definition 3.1 We say that a function $u \in L^{\vec{p}}(0, T; W_0^{1, \vec{p}}(\Omega)) \cap C(0, T; L^2(\Omega))$ is a weak solution to problem (3.1) if for all $t \in (0, T)$

$$\begin{aligned} \int_{\Omega} u(x, t) \varphi(x, t) \, dx + \int_0^t \int_{\Omega} (-u(x, \tau) \varphi_{\tau}(x, \tau) + a(x, \tau, u, \nabla u) \cdot \nabla \varphi(x, \tau)) \, dx \, d\tau \\ = \int_{\Omega} u_0(x) \varphi(x, 0) \, dx + \int_0^t \int_{\Omega} f(x, \tau) \varphi(x, \tau) \, dx \, d\tau \end{aligned} \quad (3.2)$$

for any $\varphi \in W^{1,2}(0, T; L^2(\Omega)) \cap L^{\vec{p}}(0, T; W_0^{1, \vec{p}}(\Omega))$.

Since $A(u) = -\operatorname{div}(a(x, t, u, \nabla u))$ is a pseudomonotone and coercive operator acting between $L^{\vec{p}}(0, T; W_0^{1, \vec{p}}(\Omega) \cap L^2(\Omega))$ and $\sum_{i=1}^N L^{p'_i}(0, T, W^{-1, p'_i}(\Omega) + L^2(\Omega))$, it is well-known (see [Ls] and [ACh]) that there exists a unique weak solution to problem (3.1).

Our aim is to obtain a comparison between concentrations of the solution u to problem (3.1) and the solution v to problem (1.5), which has a unique weak solution $v \in L^{\bar{p}}(0, T; W_0^{1, \bar{p}}(\Omega)) \cap C(0, T; L^2(\Omega))$.

In this section we adopt the following convention: if $h(x, t)$ is defined in Q_T , we denote by $h^*(\sigma, t)$ the decreasing rearrangement of h with respect to x for t fixed.

Theorem 3.2 Assume that (H1)-(H5) hold. Let u be the weak solution to problem (3.1) and v be the solution to problem (1.5), then we have

$$\int_0^s u^*(\sigma, t) \, d\sigma \leq \int_0^s v^*(\sigma, t) \, d\sigma \quad x \in (0, |\Omega|) \text{ for a.e. } t \in (0, T). \quad (3.3)$$

The following result is a slight extension of Theorem 3.2 when the datum in problem (1.5) is not the rearrangement of datum f of problem (3.12), but it is a function that dominates f .

Corollary 3.3 Assume the same hypothesis of Theorem 3.2. Let u be the weak solution to problem (3.1) and v be the solution to the following problem

$$\begin{cases} v_t - \operatorname{div}(\Lambda |\nabla v|^{\bar{p}-2} \nabla v) = \tilde{f}(x, t) & \text{in } Q^{\star} := \Omega^{\star} \times (0, T) \\ v(x, 0) = \tilde{u}_0(x) & \text{in } \Omega^{\star} \\ v(x, t) = 0 & \text{on } \partial\Omega \times (0, T). \end{cases} \quad (3.4)$$

where $\tilde{f} = \tilde{f}^{\star}$ and $\tilde{u}_0 = \tilde{u}_0^{\star}$ are functions such that for a.e. $t \in (0, T)$

$$\int_0^s f^*(\sigma, t) \, d\sigma \leq \int_0^s \tilde{f}^*(\sigma, t) \, d\sigma \quad \text{for } s \in [0, |\Omega|]$$

and

$$\int_0^s u_0^*(\sigma) \, d\sigma \leq \int_0^s \tilde{u}_0^*(\sigma) \, d\sigma \quad \text{for } s \in [0, |\Omega|],$$

respectively. Then we have

$$\int_0^s u^*(\sigma, t) \, d\sigma \leq \int_0^s v^*(\sigma, t) \, d\sigma \quad x \in (0, |\Omega|) \text{ for a.e. } t \in (0, T).$$

Using Corollary 3.3 it is possible to prove the following estimates of the solution $u(\cdot, t)$ to problem (3.1) in term of the solution $v(\cdot, t)$ to problem (3.4).

Corollary 3.4 *Assume the same hypothesis 3.3. If u is the weak solution to problem (3.1) and v is the solution to problem (3.4), then we have*

$$\|u(\cdot, t)\|_{L^{p,q}(\Omega)} \leq \|v(\cdot, t)\|_{L^{p,q}(\Omega^\star)} \quad \text{for } t > 0,$$

where $1 \leq p < \infty, 1 \leq q \leq \infty$ and

$$\|h\|_{L^{p,q}(\Omega)} = \begin{cases} \left[\int_0^{|\Omega|} \left(s^{\frac{1}{p}} h^{**}(s) \right)^q \frac{ds}{s} \right]^{\frac{1}{q}} & \text{if } 1 \leq p < +\infty, 1 \leq q < \infty \\ \sup_{s \in (0, |\Omega|)} s^{\frac{1}{p}} h^{**}(s) & \text{if } 1 \leq p < +\infty, q = \infty. \end{cases}$$

Let us consider a weak solution $u \in L_{loc}^{\vec{p}}(0, +\infty, W_0^{1, \vec{p}}) \cap C(0, +\infty, L^2(\Omega))$ to the following problem

$$\begin{cases} u_t - \sum_{i=1}^N \left(\alpha_i |\partial_{x_i} u|^{p_i-2} \partial_{x_i} u \right)_{x_i} = 0 & \text{in } (0, +\infty) \times \Omega \\ u(t, x) = 0 & \text{on } (0, +\infty) \times \partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega, \end{cases} \quad (3.5)$$

with $\bar{p} = 2$.

As a consequence of Theorem 3.2 we study the asymptotic behavior of solution u to problem (3.5) as time variable t goes to infinity. Proceeding as in [Ta2], it is possible to show that all the solutions to problem decay exponentially to zero as time goes to infinity.

Corollary 3.5 *Assume the same hypothesis of Theorem 3.2. If λ is the smallest eigenvalue of the following Sturm-Lionville problem*

$$\begin{cases} -\chi''(r) + \frac{n-1}{r} \chi'(r) = \lambda \chi(r) & \text{in } (0, R_\Omega) \\ \chi'(0) = \chi(R_\Omega) = 0 \end{cases} \quad (3.6)$$

and u is a non-zero solution to problem (3.5), then we have

$$\|u(t, \cdot)\|_{L^2(\Omega)} \leq e^{-\lambda t} \|u(0, \cdot)\|_{L^2(\Omega)} \quad \text{for } t > 0.$$

In order to prove Theorem 3.2 we use the well-known discretization's method. To this purpose, we divide $[0, T]$ into M subintervals

$$0 = t_0 < t_1 < \dots < t_M = T$$

with $t_{i+1} - t_i \leq \delta(M)$, where $\delta(M) \rightarrow 0$ as $M \rightarrow +\infty$. So one can approximate the solution u to problem (3.1) by the sequence $\{u_M\}_M$ of functions defined in terms of the initial datum u_0 and the weak solution to the elliptic problem

$$\begin{cases} -\operatorname{div}(a^m(x, U, \nabla U)) + \frac{U}{t_{m+1} - t_m} = f^m(x) + \frac{u^{m-1}}{t_{m+1} - t_m} & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.7)$$

where

$$\begin{aligned} a^m(x, s, \xi) &= \frac{1}{t_{m+1} - t_m} \int_{t_m}^{t_{m+1}} a(x, t, s, \xi) dt \\ f^m(x) &= \frac{1}{t_{m+1} - t_m} \int_{t_m}^{t_{m+1}} g(x, t) dt. \end{aligned}$$

More precisely,

$$u_M(x, t) = \begin{cases} u^0(x) & \text{if } t \in [0, t_1[\\ u^m(x) & \text{if } t \in [t_m, t_{m+1}[\text{ and } 1 \leq m \leq M-1, \end{cases} \quad (3.8)$$

where $u^0(x)$ coincides with $u_0(x)$ for $x \in \Omega$, and $u^m(x)$ for $1 \leq m \leq M-1$ denotes the weak solution to problem (3.7).

Analogously, the solution v to problem (1.5) can be approximated by the sequence $\{v_M\}_M$ of functions

$$v_M(x, t) = \begin{cases} v^0(x) & \text{if } t \in [0, t_1[\\ v^m(x) & \text{if } t \in [t_m, t_{m+1}[\text{ and } 1 \leq m \leq M-1, \end{cases} \quad (3.9)$$

where $v^0(x)$ agrees with $u_0^\star(x)$ for $x \in \Omega$, and $v^m(x)$ for $1 \leq m \leq M-1$ is the weak solution to the elliptic problem

$$\begin{cases} -\operatorname{div}(\Lambda |\nabla V|^{\bar{p}-2} \nabla V) + \frac{V}{t_{m+1} - t_m} = (f^m)^\star(x) + \frac{v^{m-1}}{t_{m+1} - t_m} & \text{in } \Omega^\star \\ V = 0 & \text{on } \partial\Omega^\star. \end{cases} \quad (3.10)$$

At this point to prove Theorem 3.2, we begin by checking a comparison result for elliptic problem (3.7) that we will present in the next subsection.

3.1 Comparison result for elliptic problem

In the present subsection we focus our attention to the following class of anisotropic elliptic problems

$$\begin{cases} -\operatorname{div}(a(x, w, \nabla w)) + \lambda w(x) = g(x) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.11)$$

where Ω is a bounded open subset of \mathbb{R}^N with Lipschitz continuous boundary, $N \geq 2$, $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function such that for *a.e.* $x \in \Omega$, for all $s \in \mathbb{R}^N$ and for all $\xi, \xi' \in \mathbb{R}^N$

$$(A1) \quad a(x, s, \xi) \cdot \xi \geq \sum_{i=1}^N \alpha_i |\xi_i|^{p_i} \quad \text{with } \alpha_i > 0;$$

$$(A2) \quad |a_j(x, s, \xi)| \leq \beta \left[|s|^{\bar{p}/p'_j} + |\xi_j|^{p_j-1} \right] \quad \text{with } \beta > 0 \quad \forall j = 1, \dots, N;$$

$$(A3) \quad |a_j(x, t, s, \xi) - a_j(x, t, s', \xi)| \leq \gamma |\xi_j|^{p_j-1} |s - s'| \quad \text{with } \gamma > 0 \quad \forall j = 1, \dots, N$$

$$(A4) \quad (a(x, s, \xi) - a(x, s, \xi')) \cdot (\xi - \xi') > 0 \quad \text{for } \xi \neq \xi'.$$

Moreover

(A5) $\lambda > 0$ and $g \in \left(W_0^{1, \vec{p}}(\Omega)\right)'$.

Here $1 \leq p_1, \dots, p_N < \infty$ and \bar{p} is the harmonic mean of p_1, \dots, p_N , defined in (2.4), such that $\bar{p} > 1$.

We are interested in proving a comparison result between the concentration of the solution $w \in W_0^{1, \vec{p}}(\Omega)$ to problem (3.11) and the solution $z \in W_0^{1, \bar{p}}(\Omega^\star)$ to the following problem

$$\begin{cases} -\operatorname{div}(\Lambda |\nabla z|^{\bar{p}-2} \nabla z) + \lambda z(x) = g^\star(x) & \text{in } \Omega^\star \\ z = 0 & \text{on } \partial\Omega^\star. \end{cases} \quad (3.12)$$

For this kind of results see also [AdBF2] and [AdBF3].

We emphasize that under our assumptions there exists a unique bounded weak solution (by a slight modification of classical results see *e.g.* [dC], [B] and see [ACh] as regard the uniqueness).

Theorem 3.6 *Assume that (A1)- (A5) hold. If w is the weak solution to problem (3.11) and z is the weak solution to problem (3.12), then we have*

$$\int_0^s w^*(\sigma) d\sigma \leq \int_0^s z^*(\sigma) d\sigma, \quad \forall s \in [0, |\Omega|].$$

Proof. We choose the functions $w_{\kappa, \tau} : \Omega \rightarrow \mathbb{R}$ defined as

$$w_{\kappa, \tau}(x) = \begin{cases} 0 & \text{if } |w(x)| \leq \tau, \\ (|w(x)| - \tau) \operatorname{sign}(w(x)) & \text{if } \tau < |w(x)| \leq \tau + \kappa \\ \kappa \operatorname{sign}(w(x)) & \text{if } \tau + \kappa < |w(x)| \end{cases}$$

for any fixed τ and $\kappa > 0$, as test function in problem (3.11) and by (A1), we get

$$\begin{aligned} \frac{1}{\kappa} \sum_{i=1}^N \alpha_i \int_{\tau < |w| < \tau + \kappa} \left| \frac{\partial w}{\partial x_i} \right|^{p_i} dx &\leq \frac{1}{\kappa} \int_{\tau < |w| < \tau + \kappa} a(x, w, \nabla w) dx = \\ &= \frac{1}{\kappa} \int_{\tau < |w| < \tau + \kappa} (\lambda w(x) + g(x)) (|w(x)| - \tau) \operatorname{sign}(w(x)) dx \\ &\quad + \int_{|w| > \tau + \kappa} (\lambda w(x) + g(x)) \operatorname{sign}(w(x)) dx. \end{aligned} \quad (3.13)$$

Arguing as in [C2], we can apply Polya-Szegö principle (2.12) to function $w_{\kappa, \tau}$ continued by 0 outside Ω taking into account (2.8) and (2.10). We obtain

$$\sum_{i=1}^N \alpha_i \int_{\tau < |w| < \tau + \kappa} \left| \frac{\partial w}{\partial x_i} \right|^{p_i} dx = \sum_{i=1}^N \alpha_i \int_{\mathbb{R}^N} \left| \frac{\partial w_{\kappa, \tau}}{\partial x_i} \right|^{p_i} dx \geq \Lambda \int_{\mathbb{R}^N} |\nabla w_{\kappa, \tau}^\star|^{\bar{p}} dx = \Lambda \int_{\tau < w_{\kappa, \tau}^\star < \tau + \kappa} |\nabla w_{\kappa, \tau}^\star|^{\bar{p}} dx. \quad (3.14)$$

By (3.13) and (3.14), letting $\kappa \rightarrow 0$ we get

$$-\frac{d}{d\tau} \int_{w^\star > \tau} \Lambda |\nabla w^\star|^{\bar{p}} dx \leq \int_{|w| > \tau} (|g(x)| + \lambda |w(x)|) dx \quad \text{for a.e. } \tau > 0.$$

Using Coarea formula and Hölder's inequality, we can write

$$\left(-\frac{d}{d\tau} \int_{w^\star > \tau} |\nabla w^\star|^{\bar{p}} dx \right)^{\frac{1}{\bar{p}}} \geq N \omega_N^{\frac{1}{N}} \mu_w(\tau)^{\frac{1}{N'}} (-\mu'_w(\tau))^{-\frac{1}{\bar{p}'}} \quad \text{for a.e. } \tau > 0,$$

where $\mu_w(\tau) = |\{x \in \Omega : |w(x)| > \tau\}|$. By Hardy-Littlewood inequality we obtain

$$\Lambda \left(N \omega_N^{\frac{1}{N}} \mu_w(\tau)^{\frac{1}{N'}} (-\mu'_w(\tau))^{-\frac{1}{\bar{p}'}} \right)^{\bar{p}} \leq \int_0^{\mu_w(\tau)} (\lambda w^*(s) + g^*(s)) ds \quad \text{for a.e. } \tau > 0. \quad (3.15)$$

Putting

$$\mathcal{W}(s) = \int_0^s \lambda w^*(\sigma) d\sigma \quad \text{and} \quad \mathcal{G}(s) = \int_0^s g^*(\sigma) d\sigma \quad \forall s \in [0, |\Omega|],$$

relation (3.15) gives

$$1 \leq \frac{(-\mu'_w(\tau))^{\frac{\bar{p}}{\bar{p}'}}}{\Lambda \left(N \omega_N^{\frac{1}{N}} \mu_w(\tau)^{\frac{1}{N'}} \right)^{\bar{p}}} [\mathcal{W}(\mu_w(\tau)) + \mathcal{G}(\mu_w(\tau))] \quad \text{for a.e. } \tau > 0,$$

namely,

$$1 \leq \frac{-\mu'_w(\tau) \Lambda^{-\frac{1}{\bar{p}-1}}}{\left(N \omega_N^{\frac{1}{N}} \right)^{\frac{\bar{p}}{\bar{p}-1}} (\mu_w(\tau))^{\frac{\bar{p}'}{N'}}} [\mathcal{W}(\mu_w(\tau)) + \mathcal{G}(\mu_w(\tau))]^{\frac{1}{\bar{p}-1}} \quad \text{for a.e. } \tau > 0. \quad (3.16)$$

Integrating equation (3.16) between 0 and τ , we have that

$$\tau \leq \left(N \omega_N^{\frac{1}{N}} \right)^{-\bar{p}'} \Lambda^{-\frac{1}{\bar{p}-1}} \int_{\mu_w(\tau)}^{|\Omega|} \sigma^{-\frac{\bar{p}'}{N'}} [\mathcal{W}(\sigma) + \mathcal{G}(\sigma)]^{\frac{1}{\bar{p}-1}} d\sigma \quad \text{for } \tau > 0, \quad (3.17)$$

and so

$$w^*(s) \leq \left(N \omega_N^{\frac{1}{N}} \right)^{-\bar{p}'} \Lambda^{-\frac{1}{\bar{p}-1}} \int_s^{|\Omega|} \sigma^{-\frac{\bar{p}'}{N'}} [\mathcal{W}(\sigma) + \mathcal{G}(\sigma)]^{\frac{1}{\bar{p}-1}} d\sigma \quad \text{for } s \in [0, |\Omega|]. \quad (3.18)$$

Deriving (3.18), we have that

$$(-w^*(s))' \leq \left(N \omega_N^{\frac{1}{N}} \right)^{-\bar{p}'} \Lambda^{-\frac{1}{\bar{p}-1}} s^{-\frac{\bar{p}'}{N'}} [\mathcal{W}(s) + \mathcal{G}(s)]^{\frac{1}{\bar{p}-1}} \quad \text{for a.e. } s \in [0, |\Omega|]. \quad (3.19)$$

Now let us consider problem (3.12). We recall that the solution z of (3.12) is unique and the symmetry of data assures that $z(x) = z(|x|)$, i.e. z is positive and radially symmetric. Moreover, putting $s = \omega_N |x|^N$ and $\varkappa(s) = z((s/\omega_N)^{1/N})$ we get for all $s \in [0, |\Omega|]$

$$-\Lambda |\varkappa(s)|^{\bar{p}-2} \varkappa'(s) = \frac{s^{\bar{p}/N'}}{(N \omega_N^{1/N})^{\bar{p}}} \int_0^s (\lambda \varkappa^*(\sigma) + g^*(\sigma)) d\sigma.$$

It is possible to show (see Lemma 3.2 of [FM]) that the above integral is positive and this assure that $z(x) = z^\star(x)$. By the property of z we can repeat arguments used to prove (3.19), replacing all inequalities by equalities, we obtain

$$(-z^*(s))' = \left(N \omega_N^{\frac{1}{N}} \right)^{-\bar{p}'} \Lambda^{-\frac{1}{\bar{p}-1}} s^{-\frac{\bar{p}'}{N'}} [Z(s) + \mathcal{G}(s)]^{\frac{1}{\bar{p}-1}} \quad \text{for a.e. } s \in [0, |\Omega|], \quad (3.20)$$

with

$$Z(s) = \int_0^s \lambda z^*(\sigma) d\sigma \quad \forall s \in [0, |\Omega|], \quad (3.21)$$

where z is the solution to problem (3.12). From now on, the proof is a slight modification of the proof of Theorem 5.1 of [DdB1] that we recall for the convenience of the reader. We define

$$H(s) = \int_0^s [w^*(\sigma) - z^*(\sigma)] d\sigma, \quad s \in [0, |\Omega|]$$

and we have

$$H'(|\Omega|) = 0 \text{ and } H(0) = 0.$$

We will show that

$$H(s) \leq 0 \quad \forall s \in [0, |\Omega|].$$

We proceed by contradiction and we suppose that there exists \bar{s} such that

$$H(\bar{s}) = \max_{[0, |\Omega|]} H(s) > 0.$$

We are considering two cases: $\bar{s} = |\Omega|$ and $\bar{s} < |\Omega|$.

i) If $\bar{s} = |\Omega|$, then there exists s_1 in $[0, |\Omega|]$ such that

$$H(s_1) = 0 \quad \text{and} \quad H(s) > 0 \quad \forall s \in (s_1, |\Omega|]. \quad (3.22)$$

Hence, choosing s in $(s_1, |\Omega|]$ and using (3.19) and (3.20), we get

$$\begin{aligned} w^*(s) &= - \int_s^{|\Omega|} \frac{d}{d\sigma} w^*(\sigma) d\sigma \leq \left(n\omega_n^{1/n} \right)^{-\bar{p}'} \int_s^{|\Omega|} \sigma^{-\frac{\bar{p}'}{n'}} \left(\int_0^\sigma [g^*(\tau) - \lambda w^*(\tau)] d\tau \right)^{\frac{\bar{p}'}{\bar{p}}} d\sigma \\ &< \left(n\omega_n^{1/n} \right)^{-\bar{p}'} \int_s^{|\Omega|} \sigma^{-\frac{\bar{p}'}{n'}} \left(\int_0^\sigma [g^*(\tau) - \lambda z^*(\tau)] d\tau \right)^{\frac{\bar{p}'}{\bar{p}}} d\sigma = - \int_s^{|\Omega|} \frac{d}{d\sigma} z^*(\sigma) d\sigma = z^*(s) \end{aligned}$$

in contrast to (3.22).

ii) If $\bar{s} < |\Omega|$, there exist $s_1, s_2 \in [0, |\Omega|]$ such that

$$H(s_1) = 0, \quad H(s) > 0 \quad \text{in } (s_1, s_2) \quad \text{and} \quad H'(s_2) \leq 0. \quad (3.23)$$

Hence, choosing s in (s_1, s_2) and using (3.19) and (3.20), we obtain

$$\begin{aligned} w^*(s) - w^*(s_2) &= - \int_{s_2}^s \frac{d}{d\sigma} w^*(\sigma) d\sigma \leq \left(n\omega_n^{1/n} \right)^{-\bar{p}'} \int_{s_2}^s \sigma^{-\frac{\bar{p}'}{n'}} \left(\int_0^\sigma [g^*(\tau) - \lambda w^*(\tau)] d\tau \right)^{\frac{\bar{p}'}{\bar{p}}} d\sigma \\ &< \left(n\omega_n^{1/n} \right)^{-\bar{p}'} \int_{s_2}^s \sigma^{-\frac{\bar{p}'}{n'}} \left(\int_0^\sigma [g^*(\tau) - \lambda z^*(\tau)] d\tau \right)^{\frac{\bar{p}'}{\bar{p}}} d\sigma = - \int_{s_2}^s \frac{d}{d\sigma} z^*(\sigma) d\sigma = z^*(s) - z^*(s_2), \end{aligned}$$

and being $H'(s_2) = w^*(s_2) - z^*(s_2) \leq 0$, we get

$$w^*(s) < z^*(s) \quad \text{in } (s_1, s_2)$$

in contrast to (3.23). ■

We are interested in a slight extension of Theorem 3.6 when the datum in problem (3.12) is not the rearrangement of datum g of problem (3.11), but it is a function that dominates g .

Corollary 3.7 Assume the same hypothesis of Theorem 3.6. Let z be the solution to the following problem

$$\begin{cases} -\operatorname{div}(\Lambda|\nabla z|^{\bar{p}-2}\nabla k) + \lambda z(x) = \tilde{g}(x) & \text{in } \Omega^\star \\ z = 0 & \text{on } \partial\Omega^\star, \end{cases}$$

where $\tilde{g} = \tilde{g}^\star$ is a function such that

$$\int_0^s g^*(\sigma) d\sigma \leq \int_0^s \tilde{g}^*(\sigma) d\sigma \quad \text{for } s \in [0, |\Omega|].$$

Then we have

$$\int_0^s w^*(\sigma) d\sigma \leq \int_0^s z^*(\sigma) d\sigma \quad \text{for } s \in [0, |\Omega|].$$

Proof. The result follows reasoning as in the proof of Theorem 3.6. In this case, instead of (3.16) we get

$$1 \leq \frac{-\mu'_w(t)\Lambda^{\frac{1}{\bar{p}-1}}}{\left(N\omega_N^{\frac{1}{N}}\right)^{\frac{\bar{p}}{\bar{p}-1}}(\mu_w(t))^{\frac{\bar{p}'}{N'}}} [\mathcal{W}(\mu_w(t)) + \mathcal{G}(\mu_w(t))]^{\frac{1}{\bar{p}-1}} \quad \text{for a.e. } t > 0,$$

with $\mathcal{G}(s) = \int_0^s \tilde{g}^*(\sigma) d\sigma$. ■

3.2 Proof of Theorem 3.2

Now we are in position to prove Theorem 3.2. We split the proof in three steps using the notation introduced after Corollary 3.5.

Step 1. (A priori estimate) We want to obtain the following a priori estimate

$$\sup_{[0,T]} \int_{\Omega} |u_M|^2 dx + \sum_{i=1}^N \alpha_i \int_0^T \int_{\Omega} |(u_M)_{x_i}|^{p_i} dx dt \leq C, \quad (3.24)$$

for some constant C depending only on the data.

Let us consider u^m as test function in problem (3.7). It follows that

$$\frac{1}{t_{m+1} - t_m} \int_{\Omega} (|u^m|^2 - u^m u^{m-1}) dx + \int_{\Omega} a^m(x, Du^m) \cdot Du^m dx = \int_{\Omega} f^m u^m dx.$$

Using (A1), we get

$$\frac{1}{2} \int_{\Omega} (|u^m|^2 - |u^{m-1}|^2 + |u^m - u^{m-1}|^2) dx + (t_{m+1} - t_m) \sum_{i=1}^N \alpha_i \int_{\Omega} |u_{x_i}^m|^{p_i} dx \leq (t_{m+1} - t_m) \int_{\Omega} |f^m| |u^m| dx.$$

Summing on m , we obtain

$$\frac{1}{2} \int_{\Omega} |u_M(\bar{t}, x)|^2 dx + \sum_{i=1}^N \alpha_i \int_0^{\bar{t}} \int_{\Omega} |(u_M)_{x_i}|^{p_i} dx dt \leq \int_0^{\bar{t}} \int_{\Omega} |f_M| |u_M| dx dt + \int_{\Omega} |u_0|^2 dx. \quad (3.25)$$

We estimate the right hand side of (3.25) using Hölder inequality and Young inequality

$$\begin{aligned} \int_0^{\bar{t}} \int_{\Omega} |f_M| |u_M| \, dx \, dt &\leq \int_0^{\bar{t}} \|f_M\|_{(W_0^{1,\vec{p}}(\Omega))'} \|u_M\|_{W_0^{1,\vec{p}}(\Omega)} \, dt \leq C \sum_{i=1}^N \int_0^{\bar{t}} \|f_M\|_{(W_0^{1,\vec{p}}(\Omega))'} \|\alpha_i\| (u_M)_{x_i} \|_{L^{p_i}(\Omega)} \, dt \\ &\leq C(\varepsilon) \sum_{i=1}^N \int_0^{\bar{t}} \|f_M\|_{(W_0^{1,\vec{p}}(\Omega))'}^{p'_i} \, dt + \varepsilon \sum_{i=1}^N \alpha_i \int_0^{\bar{t}} \int_{\Omega} |(u_M)_{x_i}|^{p_i} \, dx \, dt. \end{aligned}$$

Taking ε small enough and the supremum on t , we get (3.24).

Step 2. We prove that

$$\int_0^s (u^m)^*(\sigma, t) \, d\sigma \leq \int_0^s (v^m)^*(\sigma, t) \, d\sigma, \quad (3.26)$$

where u^m and v^m are the solutions of problems (3.7) and (3.10), respectively.

We proceed by induction on m .

For $m = 1$, by Lemma 2.1 we have that

$$\int_0^s \left(f^1 + \frac{u^0}{t_1 - t_0} \right)^* (\sigma) \, d\sigma \leq \int_0^s (f^1)^*(\sigma) \, d\sigma + \int_0^s \frac{(u^0)^*(\sigma)}{t_1 - t_0} \, d\sigma,$$

and then, by Corollary 3.7 it follows that

$$\int_0^s (u^1)^*(\sigma) \, d\sigma \leq \int_0^s (v^1)^*(\sigma) \, d\sigma, \quad s \in [0, |\Omega|].$$

Now assuming (3.26) to hold for $m = \nu - 1$, we will prove it for $m = \nu$.

Using Lemma 2.1 and the induction hypothesis, we get

$$\begin{aligned} \int_0^s \left(f^\nu + \frac{u^{\nu-1}}{t_{\nu-1} - t_{\nu-2}} \right)^* (\sigma) \, d\sigma &\leq \int_0^s (f^\nu)^*(\sigma) \, d\sigma + \int_0^s \frac{(u^{\nu-1})^*(\sigma)}{t_{\nu-1} - t_{\nu-2}} \, d\sigma \\ &\leq \int_0^s (f^\nu)^*(\sigma) \, d\sigma + \int_0^s \frac{(v^{\nu-1})^*(\sigma)}{t_{\nu-1} - t_{\nu-2}} \, d\sigma. \end{aligned}$$

So applying Corollary 3.7 we get (3.26) for $m = \nu$.

Step 3 (Passing to the limit). Inequality (3.26) can be written as

$$\int_0^s (u_M)^*(\sigma, t) \, d\sigma \leq \int_0^s (v_M)^*(\sigma, t) \, d\sigma, \quad (3.27)$$

for $t \in [0, T]$, where u_M and v_M are defined by (3.8) and (3.9), respectively.

To conclude, after extracting a subsequence, the estimates (3.24) yield

$$\begin{aligned} u_M &\rightharpoonup u \quad \text{weakly in } L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega)), \\ v_M &\rightharpoonup v \quad \text{weakly in } L^p(0, T; W_0^{1,p}(\Omega^\star)), \\ u_M &\overset{*}{\rightharpoonup} u \quad \text{weakly } * \text{ in } L^\infty(0, T; L^2(\Omega)), \\ v_M &\overset{*}{\rightharpoonup} v \quad \text{weakly } * \text{ in } L^\infty(0, T; L^2(\Omega^\star)), \end{aligned}$$

where u and v are the solutions to problems (3.1) and (1.5), respectively. Note that the last assertion is a consequence of classical results contained in [Ls]) thanks to which we are able to pass to the limit in (3.27) as $M \rightarrow +\infty$ and conclude the proof. ■

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